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THE EFFECT OF NON-SPHERICAL SHAPE ON THE MOTION OF A RISING UNDERWATER GAS BUBBLE

by Max Shiffman

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A Memorandum Submitted
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THE EFFECT OF NON-SPHERICAL SHAPE ON THE MOTION OF A RISING UNDERWATER GAS BUBBLE

(Preliminary Memorandum)

by Max Shiffman

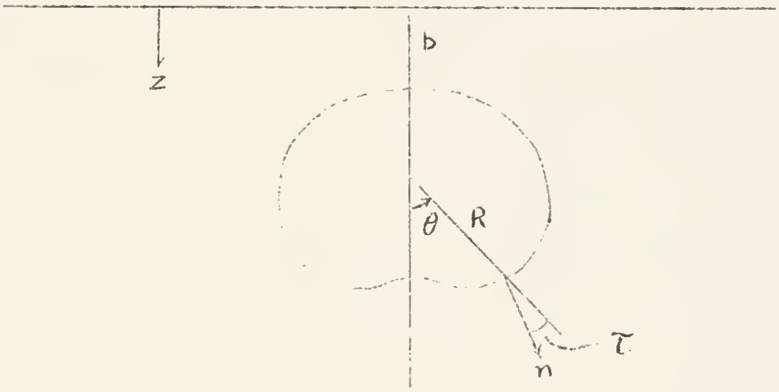
1. Introduction

The problem under consideration is the motion of a pulsating gas bubble, produced by an underwater explosion, as influenced by the buoyant force due to gravitation. This problem was first considered by C. Herring, using a perturbation method. It was then taken up by G. I. Taylor, who integrated the exact equations under the assumption that the bubble remains spherical in shape. In the present work, this assumption is weakened by allowing deviations from spherical symmetry but supposing them to be small. But, in contrast to other works on non-spherical shape, we do not assume that the velocity of rise of the bubble is small.

Experiment shows some deviations from Taylor's results. It is found that the high pressure pulse produced when the bubble is contracting and near its minimum size is larger than that predicted by Taylor's theory. Here, qualitative indications are given that non-sphericity may be the cause of these deviations. Exact statements require numerical integration of the equations.

2. The mathematical formulation of the problem.

The surface of the bubble, which is rotationally symmetric about a vertical axis, will be described by its equation $R = R(\theta)$ in polar coordinates with a specific point inside the bubble as pole. Determine this point by requiring that the



expansion of $R(\theta)$ Legendre polynomials contain no $P_1(\cos\theta)$ term, i.e.,

$$\int_0^{2\pi} R(\theta)\cos\theta\sin\theta\,d\theta = 0$$

It is easy to show that such a point exists, and is in fact unique if the surface does not deviate markedly from the spherical shape. The equation of the surface is

$$(1) \qquad R = a(1 + \lambda) \qquad \text{where} \qquad \lambda = \sum_{n=2}^{\infty} \alpha_n P_n(\cos\theta)$$

Let b be the distance of the pole from a point 33 feet above the level of the water. The quantities a, b, $\alpha_2, \alpha_3, \ldots$ are functions of t to be determined.

The advantage of this choice of the pole will appear later. Here, let it be mentioned that if one retains only first powers of the α's and neglects product terms, the volume of the bubble is

$$(2) \qquad V = \frac{4}{3}\pi a^3 \; ,$$

and the center of gravity of the bubble is at the pole.

If a, b, $\alpha_2, \ldots, \alpha_n, \ldots\ldots$ are given functions of t, the hydrodynamical state of the water at any instant t is determined by a velocity potential φ (we are assuming incompressibility of the water) which satisfies Laplace's equation, is regular at ∞, and satisfies the boundary condition:

$-\frac{\partial \varphi}{\partial n}$ = normal component of velocity of the surface at the surface of the bubble. Letting τ be the angle between the radius vector and the normal to the surface, so that

$$(3) \qquad \tan\tau = \frac{dR}{R\,d\theta} = \frac{a}{R}\,\frac{d\lambda}{d\theta} \; ,$$

the boundary condition becomes

$$-\cos\tau\frac{\partial\varphi}{\partial r} + \sin\tau\,\frac{\partial\varphi}{r\,\partial\theta} = \dot{R}\cos\tau + \dot{b}\cos(\theta-\tau)$$

or

$$(4) \qquad -\frac{\partial\varphi}{\partial r} + \tan\tau\,\frac{\partial\varphi}{r\,\partial\theta} = \dot{R} + \dot{b}\cos\theta + \dot{b}\sin\theta\tan\tau$$

on the surface of the bubble. (The dot denotes differentiation with respect to t.)

In the next section a method of successive approximations will be given for determining the potential function satisfying the boundary condition (4). Once determined, the motion of the gas bubble is found by setting the pressure of the water at the surface of the bubble equal to the gas pressure. The pressure P of the water is given by the Bernouilli equation

$$(5) \qquad \frac{P}{\rho} - g\,z + \frac{1}{2}\,(\operatorname{grad}\ \varphi)^2 - \left(\frac{\partial \varphi}{\partial t} - \dot{b}\,\frac{\partial \varphi}{\partial z}\right) = 0$$

where ρ is the density of the water, g is the acceleration due to gravity, and the extra term $\dot{b}\,\dfrac{\partial \varphi}{\partial z}$ occurs because of the moving coordinate system used for φ.

3. The determination of the velocity potential.

To determine the potential function φ satisfying the boundary condition (4), set

$$(6) \qquad \varphi = \dot{a}\,\varphi_0 + \dot{b}\,\varphi_1 + \sum_{n=2}^{\infty} (a\alpha_n)^{\cdot}\,\varphi_n \qquad ,$$

where $\varphi_0,\ \varphi_1,\ \varphi_2, \ldots\ldots,\ \varphi_n, \ldots\ldots$ are potential functions not depending on $\dot{a}$, $\dot{b}$, $(a\alpha_n)^{\cdot}$. Substituting (4), and using (3), the boundary conditions are

$$(7) \qquad \begin{aligned} -\frac{\partial \varphi_0}{\partial r} + \frac{1}{1+\lambda}\,\frac{d\lambda}{d\theta}\,\frac{\partial \varphi_0}{r\,\partial \theta} &= 1 \\ -\frac{\partial \varphi_1}{\partial r} + \frac{1}{1+\lambda}\,\frac{d\lambda}{\partial\theta}\,\frac{\partial \varphi_1}{r\,\partial \theta} &= \cos\theta + \sin\theta\,\frac{1}{1+\lambda}\,\frac{d\lambda}{d\theta} \\ -\frac{\partial \varphi_n}{\partial r} + \frac{1}{1+\lambda}\,\frac{d\lambda}{d\theta}\,\frac{\partial \varphi_n}{r\,\partial \theta} &= P_n(\cos\theta) \qquad , \quad n \geq 2 \end{aligned}$$

The functions $\varphi_0,\ \varphi_1, \ldots\ldots,\ \varphi_n, \ldots\ldots$ can be found by success-

ive approximation, assuming that the quantities $\alpha_2, \alpha_3, \ldots, \alpha_n, \ldots$ are small compared to 1 and retaining successively the zero$^{\underline{th}}$, first second, etc. order term in the α's. In the end, only the first order approximation will be used.

The 0$^{\underline{th}}$ order approximation for $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ are easily found, and we can set

$$(8) \quad \begin{cases} \varphi_0 = \dfrac{a^2}{r} + \psi_0 \\ \varphi_1 = \dfrac{a^3}{\alpha} \dfrac{\cos\theta}{r^2} + \psi_1 \\ \varphi_n = \dfrac{a^{n+2}}{n+1} \dfrac{P_n(\cos\theta)}{r^{n+1}} + \psi_n , \quad n \geq 2 \end{cases}$$

where $\psi_0, \psi_1, \ldots, \psi_n, \ldots$ are first order terms. Substituting (7), and retaining only first order terms, one obtains

$$(9) \quad \begin{cases} -\dfrac{\partial \psi_0}{\partial r}\Big]_a = 2\lambda \\ -\dfrac{\partial \psi_1}{\partial r}\Big]_a = 3\lambda\cos\theta + \dfrac{3}{2}\dfrac{d\lambda}{d\theta}\sin\theta \\ -\dfrac{\partial \psi_n}{\partial r}\Big]_a = (n+2) \ P_n(\cos\theta) + \dfrac{1}{n+1}\dfrac{d\lambda}{d\theta}\sin\theta \ P_n^1(\cos\theta) \, n \geq 2 \end{cases}$$

The solutions of (9) are

$$(10) \quad \begin{cases} \psi_0 = \displaystyle\sum_2^\infty \frac{2}{n+1} \alpha_n \frac{a^{n+2} P_n(\cos\theta)}{r^{n+1}} \\ \psi_1 = \displaystyle\sum_1^\infty \frac{3n}{2} \left(\frac{\alpha_{n-1}}{2n-1} - \frac{\alpha_{n+1}}{2n+3} \right) \frac{a^{n+2} P_n(\cos\theta)}{r^{n+1}} \end{cases}$$

The expression for ψ_n, $n \geq 2$, is too complicated to set down. But note in (6) that φ_n is required only to a zero$^{\underline{th}}$ order

since it is multiplied by $(a\alpha_n)^{\bullet}$, which is of the first order compared to $\dot{b}$. This follows from the fact that $(a\alpha_n)^{\bullet}$ is significant only when the bubble is contracting and near its minimum size. During this stage the term $a\dot{\alpha}_n$ is $(a\alpha_n)^{\bullet} = \dot{a}\alpha_n + a\dot{\alpha}_n$ is small compared to $\dot{b}$ because a is small.

Substituting (8) and (6),

$$(11) \qquad \varphi = \sum_{0}^{\infty} A_n a^{n+2} \frac{P_n(\cos\theta)}{r^{n+1}}$$

where

$$(12) \quad \begin{cases} A_0 = \dot{a} \\ A_1 = \dfrac{\dot{b}}{2} - \dfrac{3}{10}\alpha_2 \dot{b} \\ A_n = \dfrac{1}{n+1} \quad (a\alpha_n)^{\bullet} + 2\alpha_n \dot{a} \quad + \dfrac{3r}{2}\dot{b} \dfrac{\alpha_n^{-1}}{2n-1} - \dfrac{\alpha_n^{+1}}{2n+3} \quad n \geq 2 \end{cases}$$

The potential function φ can be obtained to a second or highter order, but it is useless to do so.

4. The equations of motion.

The motion of the bubble is now obtained by substituting the expression for φ in (5), evaluating at the surface of the bubble, and retaining only first order terms. Expanding (5) into a series of Legendre polynomials and equating the coefficients to zero, the result is a series of equations of which the first is simple enough to write down. It is

$$(13) \qquad \frac{P}{\rho} - g\,b + \frac{1}{4}\,\dot{b}^2 + \frac{1}{2}\,\dot{a}^2 - \frac{\dot{(a^2\,\dot{a})}}{a} = 0$$

where P is the pressure of the gas, depending on a according to the adiabatic law. This equation does not depend on the α's (to the first order) and will be interpreted in the next section.

The remaining equations are too complicated to discuss. They become simpler if we suppose that the equation (1) of the surface of the bubble is limited to

$$(14) \qquad \lambda = \alpha_2\,P_2\,(\cos\theta),$$

i.e., $\alpha_3 = \alpha_4 = \ldots\ldots = 0$. In that case, the next two equations are to the first order,

$$(15) \qquad \frac{\dot{(a^3\,\dot{b})}}{a^2} + 2\,g\,a\,(1 + \frac{9}{5}\,\alpha_2) - \frac{9}{5}\,a\,\dot{b}\,\dot{\alpha}_2 = 0$$

$$(16) \qquad a^2\,\ddot{\alpha}_2 + 5\,a\,\dot{a}\,\dot{\alpha}_2 + (3\,\dot{a}^2 - \frac{162}{35}\,\dot{b}^2)\,\alpha_2 = -\,\frac{9}{4}\,\dot{b}^2$$

The three equations (13), (15), (16) must be solved numerically.

It is convenient to set down here the expressions for the kinetic energy T of the water. Neglecting quadratic terms in α_n, $n = 2, 3, \ldots\ldots$, an elementary calculation yields

$$(17) \qquad T = 2\pi\,\rho\,a^3 \left[\,\dot{a}^2 + \frac{1}{6}\,\dot{b}^2\,(1 - \frac{9}{5}\,\alpha_2)\right] + \text{quadratic terms.}$$

(This expression for T can not be used to derive the equations of motion in the Lagrange form, since the quadratic terms are necessary for that.)

5. Discussions of the equations.

In this section we shall study qualitatively the effects of the deviations from spherical shape on the upward motion of the bubble.

a) The sign of α_2. Experimental evidence and some theoretical graphs of Penney show that, in the crucial phase when the bubble is contracting and approaching its minimum size, the quantities $\alpha_2, \dot{\alpha}_2$ become negative.

b) The upward velocity. Writing equation (15) as

$$\frac{(a^3 \dot{b})^{\cdot}}{a^2} = -2ga - \frac{18}{5} g a \alpha_2 + \frac{9}{5} a \dot{b} \dot{\alpha}_2 ,$$

the two terms $\frac{18}{5} g a \alpha_2$ and $\frac{9}{5} a \dot{b} \dot{\alpha}_2$ are positive ($\dot{b}$ is negative), so that $\frac{(a^3 \dot{b})^{\cdot}}{a^2}$ is not so large negatively (and may become positive as in the spherically symmetric one. Thus, the bubble does not rise as rapidly as predicted by Taylor's theory.

c) The minimum size. If one groups $\frac{1}{4} \dot{b}^2$ in equation (13) with the equation (13) can be interpreted as follows: in addition to the gas pressure P opposing the contraction of the bubble there is an additional opposition represented by the term $\frac{1}{4} \rho \dot{b}^2$. Now we have seen that $\dot{b}^2$ is smaller than that predicted in the spherical case. Therefore the opposition $\frac{1}{4} \rho \dot{b}^2$ to contraction is smaller than in the spherical case, and the bubble

will reach a smaller minimum size. Thus, the effect of deviations from spherical symmetry is to cause the gas bubble to contract to a smaller size than that predicted by Taylor's theory.

d) The pressure at a fixed point in the water. The pressure at a fixed point in the water can be obtained by substituting (11), (12), into (5). The result is an expression for P in powers of $\frac{a}{r}$. We are interested in the pressure when the bubble is contracting and near its minimum size, i.e., when a is small, while the point at which the pressure is derived has a relatively large distance from the bubble. The dominant term in P is therefore the first power of $\frac{a}{r}$, ~~and we~~ have

$$\frac{P}{\rho} - g\,z = \frac{(a^2\dot{a})^{\cdot}}{r} + \text{terms } \frac{1}{r^2}, \;\ldots\ldots$$

From (13)

$$(18)\quad \frac{P}{\rho} = \frac{a}{r}\left\{\frac{P(a)}{\rho} + \frac{1}{4}\dot{b}^2 + \frac{1}{2}\dot{a}^2 \quad + \right\} \;\ldots\ldots\ldots$$

where P(a) is the gas pressure inside the bubble, and hydrostatic terms have been ignored in (18).

Now, the maximum pressure at a specified point in the water will occur at or very near to the time when the bubble has its minimum size, which occurs when $\dot{a} = 0$.

At minimum size

$$(19) \qquad \frac{P}{\rho} = \frac{\bar{a}}{r}\left\{\frac{P(\bar{a})}{\rho} + \frac{1}{4}\dot{b}^2\right\}$$

where $\bar{a}$ is the minimum radius of the bubble. The pressure is compounded out of two terms - the internal pressure $P(\bar{a})$ of the gas and the dynamic term $\frac{1}{4}\rho\dot{b}^2$. It is necessary to determine which of these two terms is the most important.

The energy equation is

$$(20) \qquad T + G(a) + \frac{4}{3}\pi a^3 \rho g b = W$$

where $G(a)$ is the internal energy of the gas, W is a constant, and $\frac{4}{3}\pi a^3 \rho g b$ is the work required to expand the bubble against the hydrostatic pressure $\rho g b$. Neglecting hydrostatic terms and using (17), we have

$$(21) \qquad \dot{b}^2 = \frac{W - G(\bar{a})}{\frac{1}{3}\pi\bar{a}^3(1-\frac{9}{5}\alpha_2)}$$

when $\dot{a} = 0$.

Substituting (19), and using $G(a) = \dfrac{P(\bar{a}) \cdot \frac{4}{3}\bar{a}^3}{\gamma - 1}$

where $P(\bar{a})$ is given by the relation $P(\bar{a}) = \dfrac{\text{constant}}{\bar{a}^{3\gamma}}$,

we obtain

$$(22) \qquad P = \frac{1}{r}\left[\frac{W}{\frac{4}{3}\pi\bar{a}^2(1-\frac{9}{5}\alpha_2} - \left\{\frac{1}{(\gamma-1)(1-\frac{9}{5}\alpha_2)} - 1\right\}\bar{a}\,P(\bar{a})\right]$$

Using equations [10], [12] in Taylor's report for T.N.T., one can determine the value of $\bar{a}$ which will maximize the co-

efficient of $\frac{1}{r}$ in (22). The result for the case $\alpha_2 = 0$ is

(23) $\bar{a} = 1.30\ M^{1/3}$

for maximum possible P, where M is the mass of the explosive. In the present case of negative α_2, the critical value of $\bar{a}$ is smaller than (23).

The critical value of $\bar{a}$ is to be compared with the minimum possible value of a which would occur if there were no vertical motion of the bubble and which is determined by $G(a) = W$. This value of a is

(24) $a = 1.25\ M^{1/3}$ where $G(a) = W$.

The minimum radius obtained by Taylor in his calculations is 3.3 time (24), or

(25) $a = 4.1\ M^{1/3}$

A comparison of (23), (24), (25) shows that the upward motion of the bubble, although causing the contraction to take place nearer the target (if the target is situated above the explosion), prevents the bubble from obtaining a much smaller radius and so produces a correspondingly weaker pressure pulse. The most important factor in obtaining a high pressure pulse is to reach as small a radius as possible.

In part (c) of this section, we showed that the effect

of deviation from spherical shape is to reach a smaller radius $\bar{a}$ than otherwise. Thus, there will be a higher pressure at a specified point in the water than that predicted by Taylor's theory.

6. Conclusion.

The deviations from Taylor's thoery noted in section 5 have been observed experimentally. Non-spherical shape of the gas bubble gives a good qualitative explanation, in the right direction, of these observations. To answer whether it gives quantitative agreement with the observations would require the numerical integration of equations (13), (15), (16).

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